Form Factors for Rotational Levels in ^{12}C, ^{16}O, ^{28}Si and ^{40}Ca

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(Z. Naturforsch. **30 a**, 1018–1022 [1975]; received May 22, 1975)

We support the proposed ground state rotational bands in ^{12}C, ^{16}O, ^{28}Si and ^{40}Ca by calculating the form factors for the electro-excitation within a simple model. The model introduces two parameters for the size and the surface thickness of these nuclei and allows for the prediction of the transition probabilities and form factors within the rotator model. The model unambiguously predicts the sequence of rotational levels 0^+, 3^+, 4^+ ... for ^{16}O and ^{40}Ca, and 0^+, 2^+, 4^+ ... for ^{12}C and ^{28}Si in agreement with the experiment. The comparison of the calculated form factors with the experimental data shows excellent agreement in the cases ^{28}Si and ^{40}Ca and is less satisfactory in the cases ^{12}C and ^{16}O.

1. Introduction

Rotational bands are well-known phenomena mainly associated with the heavier, deformed nuclei, such as the rare-earth and actinide nuclei. In the light nuclei up to ^{40}Ca a few cases are well established, notably ^{20}Ne, ^{24}Mg, ^{28}Si. In other light nuclei such as the important doubly magic nuclei ^{16}O and ^{40}Ca some evidence for the existence of ground state rotational bands has been reported. ^{16}O has been discussed by Dennison [1] in 1940 in connection with the α-particle model, and more recently together with ^{40}Ca (and ^{208}Pb) by the present author [2]. In both nuclei the band consists of the sequence of levels 0^+, 3^- (6.13 MeV), 4^+ (10.4 MeV) in ^{16}O and 0^+, 3^- (3.74 MeV), 4^+ (5.28 MeV) in ^{40}Ca. This differs from the usual sequence 0^+, 2^+, 4^+, ... by the replacement of the 2^+ level by a 3^- level at somewhat higher energy. This special feature has been explained by assuming intrinsic tetrahedral deformation in place of the usual quadrupole type deformation, and has been shown in Ref. [2] to be independent of any specific nuclear model. Also in the case of ^{12}C Nakada [3] et al. have proposed to consider the levels 2^+ at 4.43 MeV and 4^+ at 14.1 MeV as the members of a rotational band based on the nuclear ground state. Thus, there is good evidence that rotational bands may in fact be general features in the physics of these nuclei. The evidence consists mainly of the energies of these levels which obey the $I(I+1)$ law for axially symmetric and spherical rotators. On the other hand the existence of these bands is masked by a large number of nuclear levels which may be interpreted as intrinsic modes and have energies of similar magnitude. This is quite in contrast to the situation in the heavy nuclei which have their rotational bands clearly separated from the intrinsic modes. The difference is due to a different A dependence of these effects. Whereas the rotational energies vary as $A^{-5/3}$ due to the variation of the moments of inertia (disregarding the well-known difficulties with the moments of inertia which lead to differences of the order of a factor of 2), the intrinsic modes in contrast vary as $A^{-2/3}$ i. e. with the nuclear surface according to the collective Bohr-Mottelson model, say.

As far as the nuclei ^{12}C, ^{16}O and ^{40}Ca are concerned more evidence for a ground state rotational band is needed both theoretical and experimental. A great deal can be done experimentally by measuring intraband transition probabilities, some unknown inelastic form factors, and also searching for the predicted higher levels. In order to give further support from the theoretical side transition probabilities and form factors should be calculated which, however, depend strongly on the model used for the nuclear charge distribution.

In the present paper we calculate the form factors for the electro-excitation of the rotational levels within a specific nuclear model which uses 2 parameters only and describes the rotational bands in all 4 nuclei: ^{12}C, ^{16}O, ^{28}Si and ^{40}Ca. In contrast to more phenomenological models we do not introduce parameters which specify the appropriate nuclear deformations such as a β_2 parameter for the quadrupole deformation. For a more extensive discussion of the models used to described these levels with interpretations other than the rotational band assignment see Ref. [2] for ^{16}O and ^{40}Ca.

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2. The Model

The model used has been proposed earlier by the author [4] and will be briefly reviewed subsequently. In the model specific geometrical positions are predicted which will figure as the centers of localized single-particle functions. The coordinates of these geometrical positions are obtained as the centers of a structure of close-packed spheres. At each center we localize a Gaussian charge distribution. Thus the model has only 2 parameters: the width b of the Gaussians and the distance d between adjacent centers. The use of Gaussians is convenient mathematically yet may not be a very good choice physically: its width parameter b should primarily be a measure of the localization of the nucleons within the model, at the same time it is used to adjust the surface thickness of the structure. We have nevertheless used the Gaussian in this first investigation mainly because it depends on 2 parameters only, and we have, in fact, kept these fixed for all 4 nuclei (a modification in case of ^{12}C is discussed below).

The precise geometrical positions can be obtained as follows: for ^{16}O and ^{40}Ca we start with a plane triangular array of spheres having 4 and 6 spheres along each side respectively. On top of this layer we keep adding layers of spheres until a tetrahedral pyramid is reached. Eventually we take away a sphere at each of the corners in case of ^{16}O, and a small tetrahedron of 4 spheres at each of the corners in case of ^{40}Ca. In order to obtain ^{12}C and ^{28}Si we start again with a triangular array of spheres (3 and 4 spheres along each side respectively), yet place further layers of spheres on both sides symmetrically. In order to associate protons and neutrons with the single-particle functions localized at these centers we note the important structural feature which has been termed local α-particle. This, like the true α-particle has nucleons associated with the four corners of a tetrahedron (2 protons and 2 neutrons with spin up and down). However, after antisymmetrizing the corresponding product of single-particle functions (each localized at a geometrical center) each position will have the same probability for finding the charge $+1$ there, so that we are not concerned with these structural details in the following.

Summarizing, the model predicts tetrahedral shapes for ^{16}O and ^{40}Ca, and hence implies the 0^+, 3^-, 4^+, ... ground state rotational band, and at the same time predicts dihedral symmetry for ^{12}C and ^{28}Si and the corresponding bands 0^+, 2^+, 4^+, ...

3. General Formalism

The cross section for the electro-excitation of a transition from a state of spin J_i to one of spin J_f is given in the Born approximation by [5]

$$\frac{d\sigma}{d\Omega} = \frac{4\pi\,\sigma_{\text{Mott}}}{1+2\,\varepsilon_1 \sin^2\tfrac{1}{2}\theta/M_T}\left\{\frac{\Delta^4}{q^4}\sum_{L=0}^{\infty}\frac{|\langle J_f \| M_L^{\text{e}}(q) \| J_i\rangle|^2}{2J_i+1} + \left(\frac{\Delta^2}{2q^2}+\tan^2\tfrac{1}{2}\theta\right)\sum_{L=1}^{\infty}\frac{1}{2J_i+1}\left[|\langle J_f \| T_L^{\text{e}}(q)\| J_i\rangle|^2 + |\langle J_f \| T_L^{\text{m}}(q)\| J_i\rangle|^2\right]\right\} \quad (1)$$

with $$\sigma_{\text{Mott}} = \alpha^2 \cos^2\tfrac{1}{2}\theta/4\,\varepsilon_1^2 \sin^4\tfrac{1}{2}\theta$$

where Δ and q are, respectively, the four-momentum and three-momentum transfer to the nucleus, and the transition operators are given by

$$M_{LM}^{C}(q) = \int \varrho(\mathbf{r})\, j_L(qr)\, Y_{LM}(\hat{\mathbf{r}})\, d^3\mathbf{r} \quad (2)$$

For electric collective transitions at moderate momentum transfer and scattering angles not close to 180° the transverse terms may be neglected and we will not consider them further here. The nuclear charge operator ϱ appearing in Eq. (2) can be decomposed into spherical tensor components:

$$\varrho = \sum_L \sqrt{2L+1}\,[\varrho^{[L]} \times Y^{[L]}]^{[0]}. \quad (3)$$

We then have

$$M^{[L]}(q) = \int_0^{\infty} \varrho^{[L]}(r)\, j_L(qr)\, d^3\mathbf{r}. \quad (4)$$

The inelastic electron scattering form factor for the excitation of the transition (L) is given by

$$F^2(q) = \frac{4\pi}{2J_i+1}\sum_L |\langle f \| M^{[L]}(q) \| i\rangle|^2. \quad (5)$$

In the present calculation the states of the nucleus are described by the rigid rotator model [6], as

$$|IMK\rangle = \sqrt{\frac{2I+1}{16\pi^2(1+\delta_{K0})}} \cdot [D_{MK}^{I*} + (-1)^I D_{M-K}^{I*}]\,|\text{int}\rangle, \quad (6)$$

where $D^I_{MK}(\alpha\beta\gamma)$ are the usual rotation matrices which depend on the three Euler angles and are characterized by the total angular momentum I and its projections on the space-fixed z-axis (M) and the intrinsic z-axis (K). $|\text{int}\rangle$ is the part of the wave function which depends on intrinsic coordinates only. In the absence of any nuclear dynamics it simply describes a rigid nuclear charge distribution. The eigenvalues corresponding to the states $|IMK\rangle$ are well-known,

$$E = \frac{\hbar^2}{2J_0}[I(I+1) - K^2] + \frac{\hbar^2}{2J_3}K^2 \quad (7)$$

for an axially symmetric rotator, and

$$E = (\hbar^2/2J_0)\, I(I+1)\,, \quad (8)$$

for a spherical rotator for which all the three moments of inertia are equal. Generally the bands are characterized by K ($\geqq 0$) and consist of the members $I > K$, $-I \leqq M \leqq I$. However, for specific intrinsic symmetries several of the predicted states will be forbidden. This is well-known in the case of quadrupole-type intrinsic deformations and yields the sequence, 0^+, 2^+, 4^+, ... forbidding odd-parity states and also the states 1^+, 3^+, ... in the ground state band. Given other intrinsic symmetries the analogous result may be obtained by group theoretical means[7, 8]. In case of intrinsic tetrahedral symmetry we have a spherical rotator and the sequence of states 0^+, 3^-, 4^+, 6^+, 6^-, ... and in case of intrinsic dihedral symmetry an axially symmetric rotator and the states 0^+, 2^+, 4^+, 6^+, ...

With the nuclear states given by the rotational model as in Eq. (6) we can calculate the form factors for specific transitions by transforming the transition operator from laboratory coordinates to Euler angles and intrinsic coordinates, explicitly.

$$M^{\text{Lab}}_{LM} = \sum_{M'} M^{\text{int}}_{LM'}\, D^{L*}_{MM'}(\alpha\beta\gamma)\,. \quad (9)$$

Using Eqs. (5), (6), (9) we obtain for the form factor for transitions from the ground state to the K-band (within the ground state band $K = 0$),

$$|F(q)|^2 = 4\pi\,|\langle \text{int}\,|\,M^{\text{int}}_{LK}(q)\,|\,\text{int}\rangle|^2\,, \quad (10)$$

where

$$M^{\text{int}}_{LK} = \int \varrho^{\text{int}}(\mathbf{r})\, Y_{LK}(\hat{\mathbf{r}})\, j_L(q\,r)\, \mathrm{d}^3\mathbf{r}\,. \quad (11)$$

For the special case of a spherical rotator all K values are degenerate (no special intrinsic z-axis can be identified) so that we have to sum over all K values in a rotational invariant manner, i. e.

$$|F(q)|^2 = 4\pi \sum_{K=-L}^{L} |\langle \text{int}\,|\,M^{\text{int}}_{LK}(q)\,|\,\text{int}\rangle|^2\,. \quad (12)$$

For intrinsic tetrahedral symmetry we have furthermore that only the terms $K = 0, \pm 3, \pm 6, \ldots$ contribute and also that

$$\langle \text{int}\,|\,M^{\text{int}}_{L-K}\,|\,\text{int}\rangle = \langle \text{int}\,|\,M^{\text{int}}_{LK}\,|\,\text{int}\rangle\,,$$

hence

$$|F(q)|^2 = 4\pi\{|\langle \text{int}\,|\,M^{\text{int}}_{L0}\,|\,\text{int}\rangle|^2 + 2\,|\langle \text{int}\,|\,M^{\text{int}}_{L3}\,|\,\text{int}\rangle|^2 + \ldots\}\,. \quad (13)$$

4. Results

In the explicit evaluation of the form factors according to Eqs. (10) and (13) we have placed Gaussians $\exp\{-r^2/2b^2\}$ of width $b = 0.80$ fm at the geometrical centers as predicted by the model, with distances $d = 2.0$ fm inbetween adjacent centers. The choice of d and b is such as to reproduce the elastic form factors up to the second maximum correctly ($d = 2.0$ fm can also be deduced directly from the known nuclear radii $1.1 \times A^{1/3}$ and the knowledge of the packing fraction of 0.74 for the structure of close-packed spheres).

The same values for the two parameters have been used for all four nuclei ^{12}C, ^{16}O, ^{28}Si and ^{40}Ca (with a slight modification in case of ^{12}C).

The results obtained for the four nuclei are shown in Figs. 1 to 4. In all cases we have inserted in the figures the elastic form factor which can be fitted well up to the second maximum. In Figs. 1 and 3 we show the results for ^{12}C and ^{28}Si. As far as ^{28}Si is concerned we find that the model fits the experimental data rather well, both the form factors for the transitions to the 2^+ level and to the 4^+ level agree rather closely with the measurements. In case of ^{12}C the fit is not very satisfactory. In fact the discrepancy between the theoretical fit and the experimental results is more than a factor of ten, in both cases, for the 2^+ and 4^+ level. We have improved the fit somewhat by a slight modification of the structure, thereby reducing the discrepancy to factors of 4 and 6. In the modified structure the layers of spheres adjacent to the central layer have been placed at a closer distance namely 1.0 fm instead of 1.6 fm. At the same time the distances within the layers have been increased to 2.4 fm.

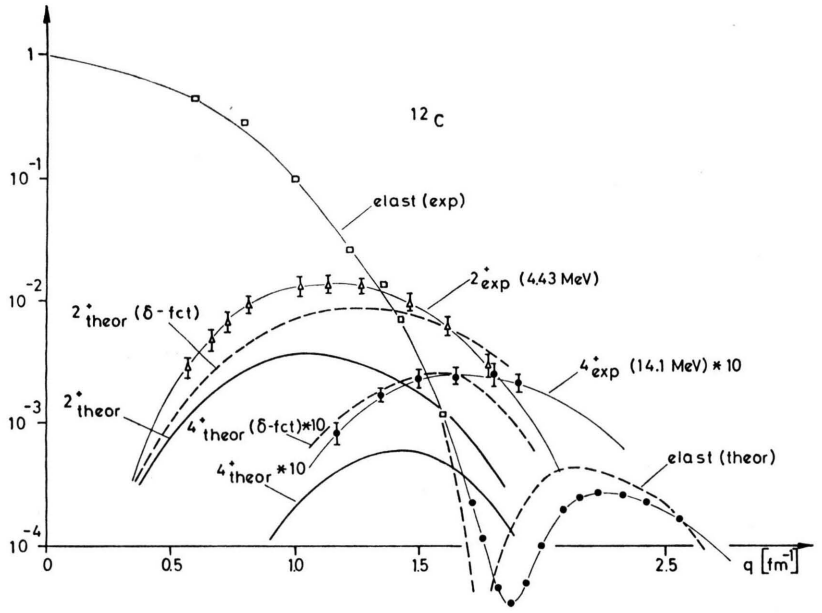

Fig. 1. Elastic and inelastic form factors for ^{12}C. The experimental data are taken from Refs. 9, 10 and 11.

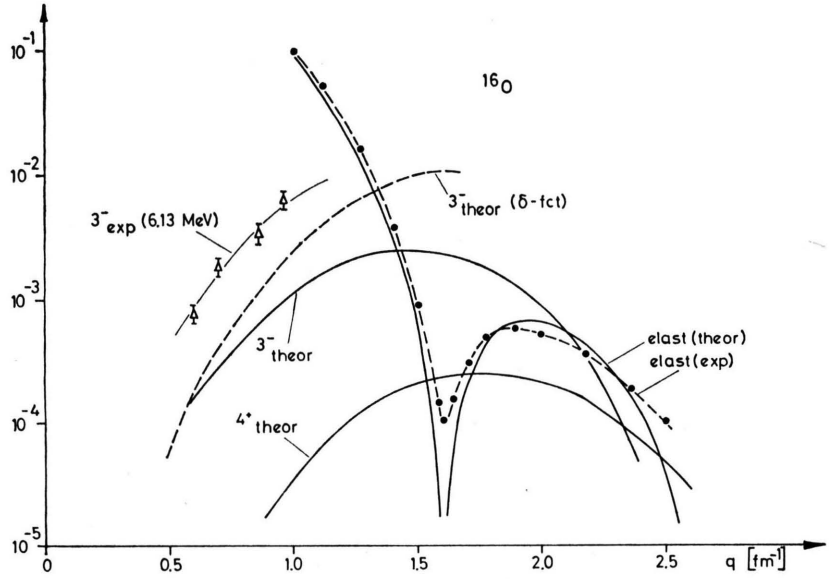

Fig. 2. Elastic and inelastic form factors for ^{16}O. The experimental data are taken from Refs. 9 and 12.

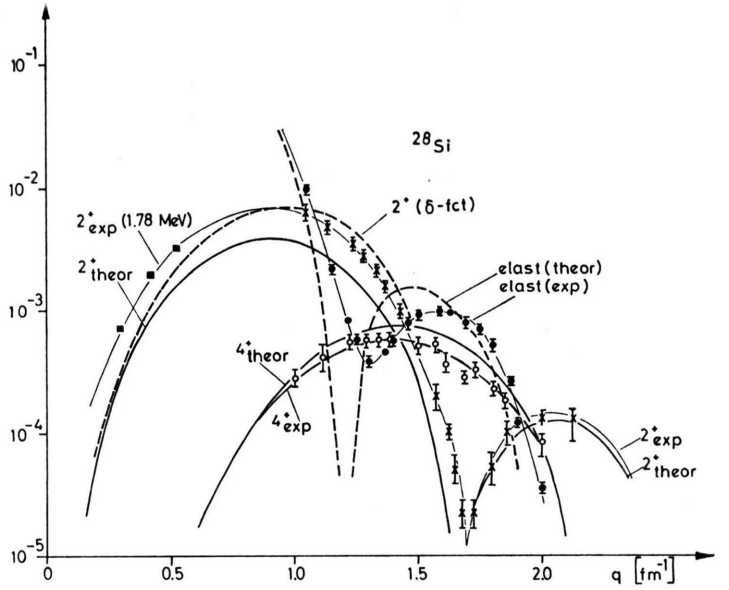

Fig. 3. Elastic and inelastic form factors for ^{28}Si. The experimental data are taken from Refs. 13 and 14.

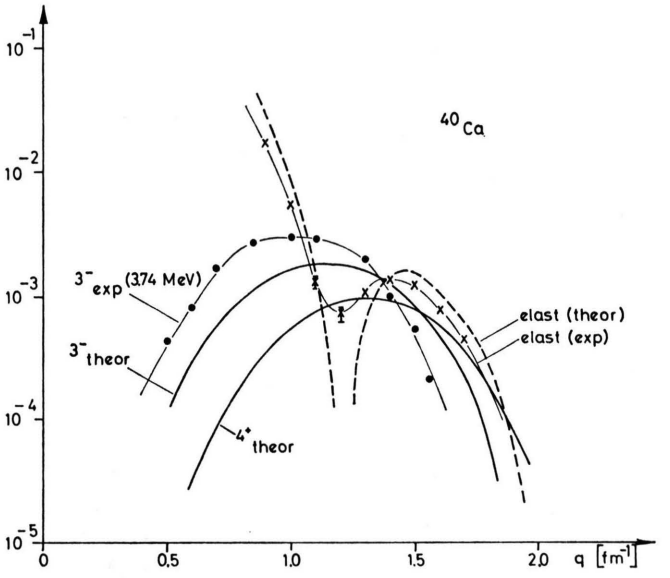

Fig. 4. Elastic and inelastic form factors for ^{40}Ca. The experimental data are taken from Refs. 15 and 16.

These changes can be justified qualitatively: since the packing of the central layer is omit hexagonal the two spheres in the two adjacent planes are separated by 3.2 fm only. Any residual attraction will result in a shortening of this distance and a simultaneous increase of the distance within the layer. For ^{28}Si the same mechanism is expected to be relevant, although somewhat weakened. However, since it turns out to be less important here for the fitting of the form factors we have neglected it.

In Figs. 2 and 4 we show the results for ^{16}O and ^{40}Ca. In both cases the experimental form factors to the 4^+ levels are not known. The agreement of the form factors to the 3^- level is excellent in the case of ^{40}Ca, and somewhat less satisfactory in the case of ^{16}O yielding a discrepancy of a factor of 4.

A possible explanation for the origin of the discrepancy may be found in the use of simple Gaussian orbitals. The large width of the Gaussians tend to make the structures more spherical and hence implies a reduction of the transition strengths. This remark is supported by the observation that the discrepancy is found in the lighter nuclei (being almost absent in ^{28}Si and ^{40}Ca). In order to study the effect we have replaced the Gaussians by δ-functions which are expected to maintain all the multipole strengths of the structures. The corresponding form factors are shown in the figures, being closer to the experimental curves than the fit with the Gaussians. A further improvement, and in fact a more realistic treatment is expected through the use of symmetry-adapted orbitals, instead of the spherical Gaussians. These functions, when appropriately defined, supposedly maintain the multipole strengths of the structures also in the outermost surface regions which will be the most important for the form factors due to the occurrence of r^L in the transition matrix elements. We have not attempted to use such functions in this first investigation mainly because we would have to introduce at least 1 or 2 new parameters.

5. Summary and Discussion

We have applied a simple model to fit the experimental form factors for the levels of a ground state rotational band in ^{12}C, ^{16}O, ^{28}Si and ^{40}Ca. The results are satisfactory in general (with the exception of ^{12}C, maybe) and further support the existence of such bands in these nuclei. The model which has been used places Gaussians of width $b = 0.80$ fm at specific geometrical positions which are obtained as the centers of the structure of close-packed spheres – adjacent centers being separated by 2.0 fm.

Thus, we have used a model which introduces 2 parameters and predicts for the ground state rotational bands unambiguously the sequence 0^+, 3^-, 4^+, 6^+, 6^-, ... in the cases ^{16}O and ^{40}Ca, and at the same time the sequence 0^+, 2^+, 4^+, 6^+, ... in the cases ^{12}C and ^{28}Si. These features are in excellent agreement with the experiment. The form factors for the transitions from the ground state to these levels have been calculated and have been found to agree well with the experiment for the heavier nuclei ^{28}Si and ^{40}Ca, and to differ by approximately a factor of 4 in the case of ^{12}C and ^{16}O. The origin of the latter discrepancy within the model has been discussed above, and may be due in part also to the neglect of exchange and correlation effects.

Summarizing we may say that it makes sense to have different rotational bands in different nuclei such as the sequence 0^+, 3^-, 4^+, ... in ^{16}O and ^{40}Ca, and 0^+, 2^+, 4^+, ... in ^{12}C and ^{28}Si. As demonstrated by the simple model this does not lead to any contradiction. Also, the fits show that the assignment of ground-state rotational bands for these nuclei can be supported by the calculated form factors.

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